Theorem 2:

Let M be a compact oriented 3-manifold without boundary. Suppose that M is obtained as Dehn surgery on a framed link L with m components L_j, $1 \le j \le m$, in S^3. Then,

$$Z_K(M) = S_{00}\, C^{\sigma(L)} \sum_{\{\lambda\}} S_{0\lambda_1} \cdots S_{0\lambda_m} J(L; \lambda_1, \dots, \lambda_m)$$

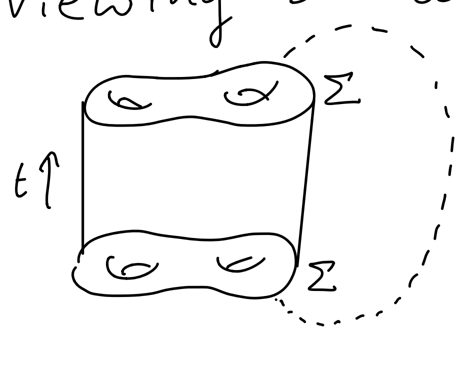
is the Chern-Simons partition funct. of M.

Let us first compute $Z_K(S^3)$ correspoding to the case with no link.

Lemma 2: $Z_K(S^3) = S_{00}$

Proof:

The first step is to compute $Z_K(\Sigma \times S^1)$. Viewing S^1 as the "time direction," we get

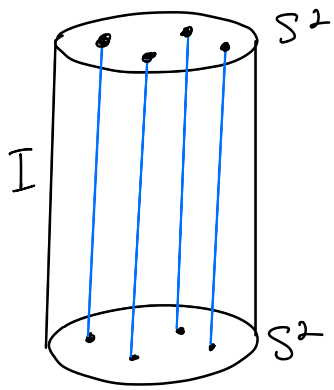


$$\to Z_K(\Sigma \times S^1) = \mathrm{Tr}_{\mathcal{H}_\Sigma}(1) = \dim \mathcal{H}_\Sigma$$

In the case of $\Sigma = S^2$, we get:

$$\dim \mathcal{H}_{S^2} = 1 \Rightarrow Z_k(S^2 \times S^1) = 1$$

In case there are Wilsonlines passing through S^2 and along S^1, we get:



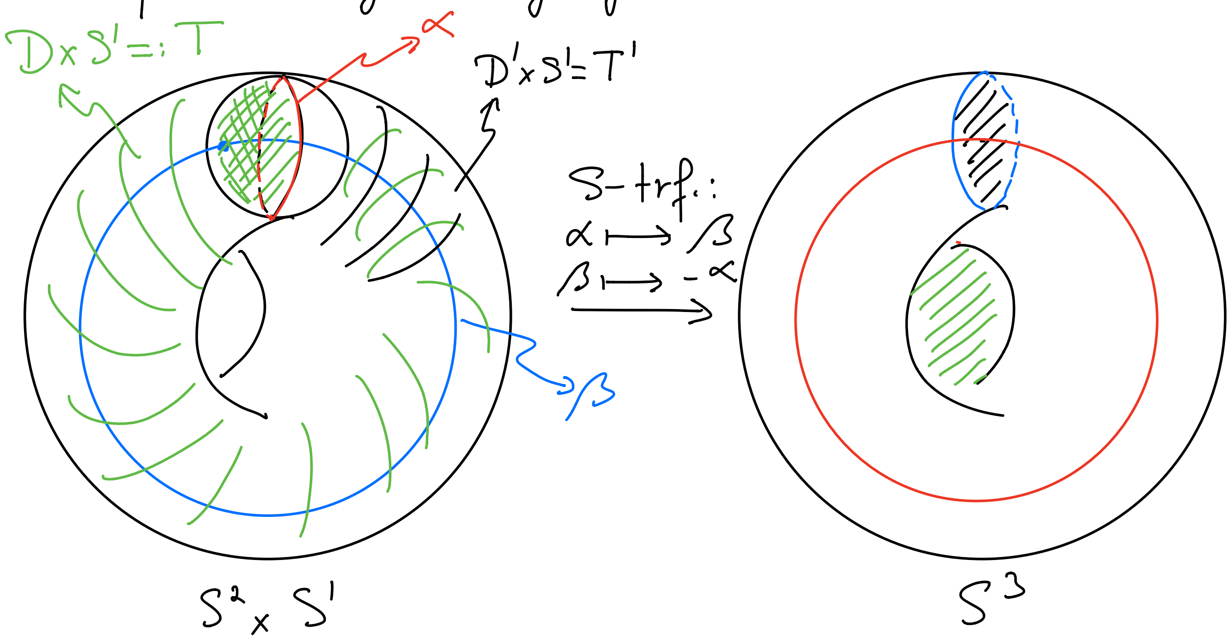
$$Z_k(S^2 \times S^1; \langle R \rangle) = \dim \mathcal{H}_{S^2; \langle R \rangle}$$

$$\rightarrow Z_k(S^2 \times S^1; R_\lambda) = \delta_{\lambda,0}$$

$$Z_k(S^2 \times S^1; R_\lambda, R_\mu) = \delta_{\mu,\lambda^*}$$

$$Z_k(S^2 \times S^1; R_\lambda, R_\mu, R_\nu) = N_{\lambda\mu\nu}$$

Now S^3 can be obtained from $S^2 \times S^1$ by the following surgery:



We see that $S^2 \times S^1$ is obtained by gluing solid tori T and T' by identifying

$$\partial T = \partial T'$$

Similarly, S^3 is obtained by identifying:

$$\partial T = \underset{\substack{\uparrow \\ S\text{-trf.}}}{S}\, \partial T'$$

At the level of conformal blocks, this gives:

$$Z_k(S^2 \times S^1; R_0) = \langle \psi | \chi_0 \rangle$$

$$\Rightarrow Z_k(S^3; R_0) = \langle \psi | S \chi_0 \rangle$$

$$= \sum_\mu \langle \psi | S_0^{\ \mu} \chi_\mu \rangle = \sum_\mu S_0^{\ \mu} \langle \psi | \chi_\mu \rangle$$

$$= \sum_\mu S_0^{\ \mu} \underbrace{Z_k(S^2 \times S^1; R_\mu)}_{= \delta_{\mu,0}} = S_{00}$$

$\square$

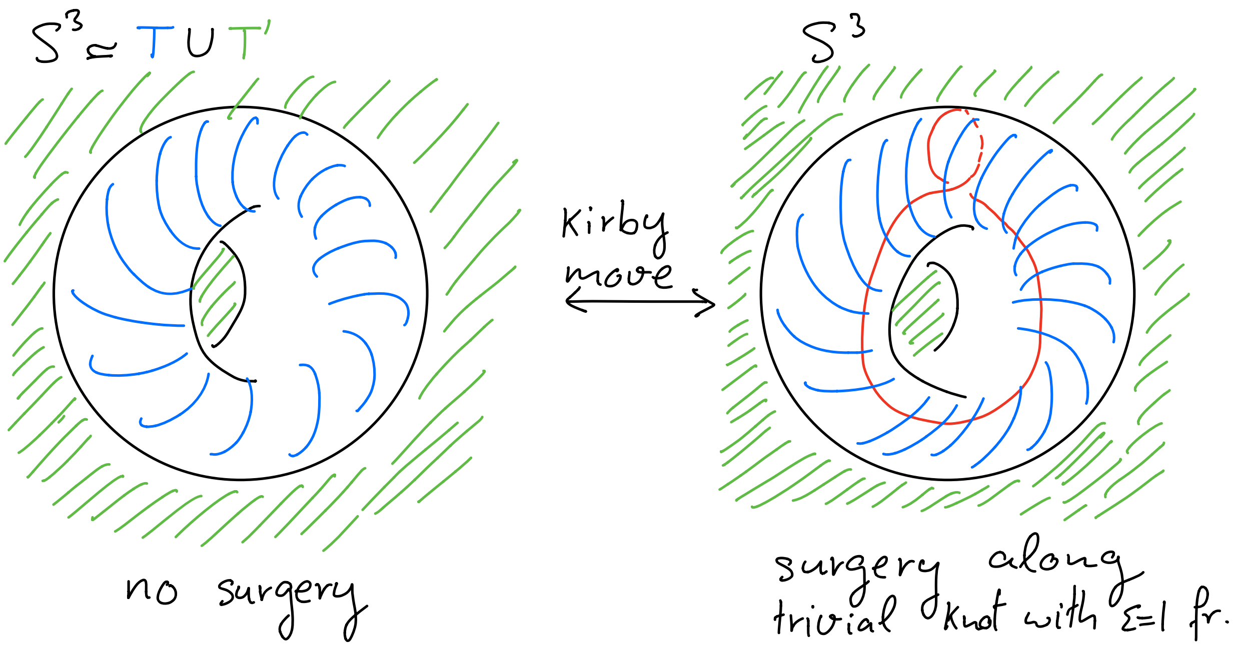
<u>Proof of Theorem 2:</u>

Let us first deal with the case <u>$n=0$</u>:

We know by Lemma 2 that $Z_k(S^3) = S_{00}$

We thus have to show:

$$S_{00}\, C \sum_{\mu \in P_+(k)} S_{0\mu}\, \mathcal{J}(O; \mu)\, e^{\underset{\substack{\uparrow \\ \text{increase of framing}}}{2\pi\sqrt{-1}\Delta_\mu}} = S_{00}$$



Now we know $J(O;\mu) = \frac{S_{0\mu}}{S_{00}}$

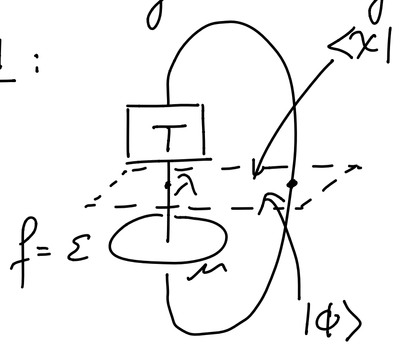
→ Lemma 1 for $\lambda = \nu = 0$ gives:

$$S_{00}\, C \sum_{\mu \in P_+(k)} S_{0\mu} \underbrace{\frac{S_{0\mu}}{S_{00}}}_{= \frac{Z_k(S^3;R_\mu)}{Z_k(S^3)}} e^{2\pi\sqrt{-1}\Delta_\mu} = S_{00} = Z_k(S^3)$$

normalization

The factor C corrects for factors due to "framing ambiguity" of 3-manifold M.

n=1:

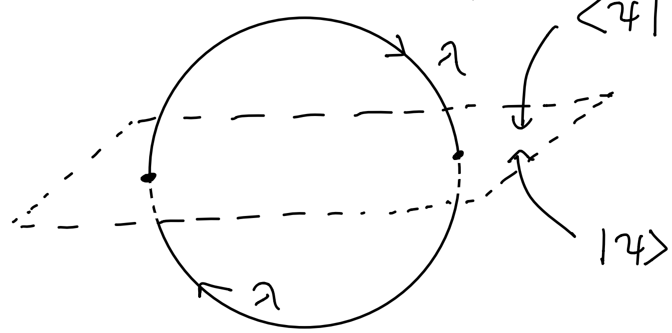


$$= J(L;\lambda,\dots)$$

$$= \frac{\langle x|\phi\rangle}{Z_k(S^3)} = \frac{1}{Z_k(S^3)} \sum_{\psi} \frac{\langle x|\psi\rangle\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}$$

$(*)$

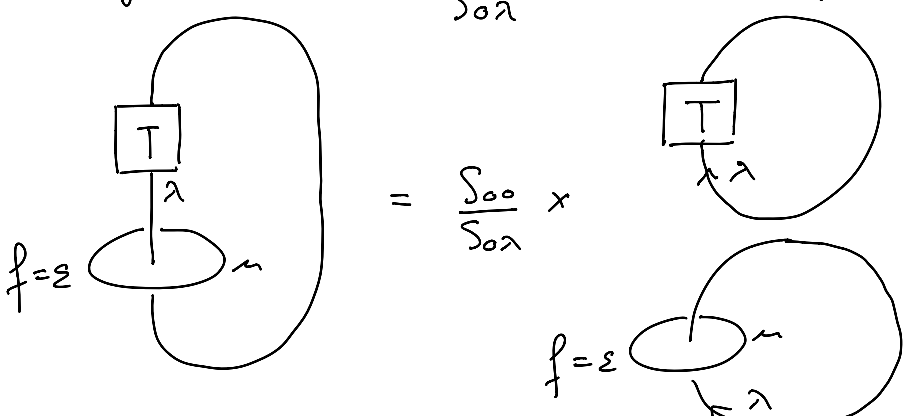
Take as basis the following ψ:



$$\longrightarrow \langle\psi|\psi\rangle = Z_k(S^3; R_\lambda) = S_{00}\underbrace{\mathcal{J}(\bigcirc; \lambda)}_{=\frac{S_{0\lambda}}{S_{00}}}$$

Inserting back into (*) gives:

$$\mathcal{J}(L; \lambda, \dots) = \frac{S_{00}}{S_{0\lambda}}\mathcal{J}(L_1; \lambda, \mu)\,\mathcal{J}(L_2; \lambda, \dots)$$



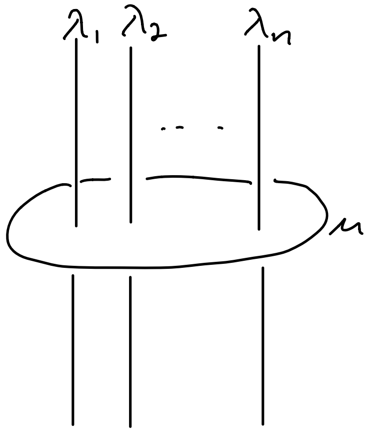
"factorization"

$\longrightarrow$ It is enough to consider the Hopf link

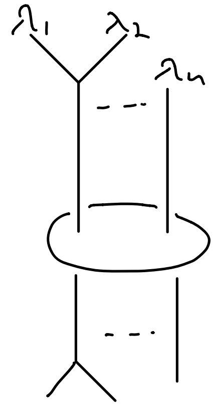
Proposition 1 + Lemma 1 for $\nu = 0$ give:

$$C\sum_{\mu\in P_+(k)} S_{0\mu}\underbrace{\frac{S_{\lambda\mu}}{S_{00}}}_{=\mathcal{J}(\bigcirc\!\!\bigcirc)}\underbrace{\exp\left(2\pi\sqrt{-1}\,\Delta_\mu\right)}_{\text{framing shift of } \bigcirc_\mu} = \underbrace{\exp\left(-2\pi\sqrt{-1}\,\Delta_\lambda\right)}_{\text{framing } = -1}\underbrace{\frac{S_{0\lambda}}{S_{00}}}_{=\mathcal{J}(\bigcirc; \lambda)}$$

Next, we show invariance of $Z_\kappa(M)$ under Kirby moves by induction on n. Consider the local situation



$$= \sum_{\nu} N^{\nu}_{\lambda_1 \lambda_2}$$



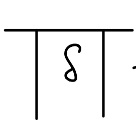
T $\qquad$ T''

It will be sufficient to show

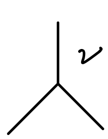
$$C^{\sigma(L)} \sum_{\mu} S_{0\mu}\, J(T; \mu, \lambda_1, \dots, \lambda_n) = C^{\sigma(L')} J(T'; \lambda_1, \dots, \lambda_n)$$

where L' is obtained from L by a Kirby move of deleting $\bigcirc^{\mu}$ and twisting by ε.

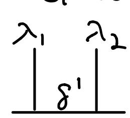
$\rightarrow$ To achieve this, fuse two strands λ_1 and λ_2 (see above) and write



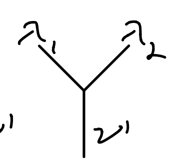
$= \sum_{\nu} F_{\delta\nu}$



and



$= \sum_{\nu'} F_{\delta'\nu'}$



$\rightarrow$ The tangle operator $J(T; \mu, \lambda_1, \dots, \lambda_n)$ is expressed as a linear combination

$$\sum_{\nu} N_{\lambda_1,\lambda_2}^{\nu} F_1 \, J(T''; \mu, \nu, \lambda_3, \dots, \lambda_n) F_2$$

where F_1 and F_2 are the above elementary connection matrices and T'' is an $(n-1, n-1)$-tangle $\longrightarrow$ have reduced situation to $n-1$ strands

By induction hypothesis and change of $\sigma(L)$ under Kirby moves, we obtain the desired statement.

$\square$

Proposition 2:

For a connected sum $M_1 \# M_2$ of closed oriented 3-manifolds M_1 and M_2

$$Z_K(M_1 \# M_2) = \frac{1}{S_{00}} Z_K(M_1) Z_K(M_2)$$

holds.

Proposition 3:

We denote by $-M$ the 3-manifold M with the orientation reversed. Then we have

$$Z_K(-M) = \overline{Z_K(M)}$$

Proof:

If M is obtained as Dehn surgery on a framed link L, then surgery on its mirror image $-L$ yields $-M$. Result follows from Prop. 3, §9.

$\square$

Extend the above construction to case where 3-manifold M contains a link L:
Let $L_1, \dots, L_n$ be components of L with coloring $\lambda_1, \dots, \lambda_n \in P_+(K)$. Suppose $(S^3, L') \longrightarrow (M, L)$ is obtained by Dehn surgery on framed link $N \subset S^3$. assume: $N \cap L' = \emptyset$. Let $N_1, \dots, N_m$ be the components of N. Then we have

$$Z_K(M, L; \lambda_1, \dots, \lambda_n)$$
$$= S_{00}\, C^{\sigma(N)} \sum_{\mu} S_{0\mu_1} \cdots S_{0\mu_m} J(L' \cup N; \lambda_1, \dots, \lambda_n; \mu_1, \dots, \mu_m)$$